

# Option-Implied Risk-Neutral Distributions and Implied Binomial Trees: A Literature Review

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*In this selective literature review, we start by observing that in efficient markets, there is information incorporated in option prices that might help us to design option pricing models. To this end, we review the numerous methods of recovering risk-neutral probability distributions from option prices at one particular time to expiration and their applications.*

*Next, we move beyond one time to expiration to the construction of implied binomial trees, which model the stochastic process of the underlying asset. Finally, we describe extensions of implied binomial trees, and other non-parametric methods.*

For a long time, option pricing has been conducted in a rather systematic way. First, one makes an assumption about the data-generating process. That is, one assumes a particular stochastic process that drives the underlying asset price. The most common choice here is the assumption of geometric Brownian motion for the asset price, first advocated by Osborne [1959]. Second, one rewrites that process in risk-neutral terms either using arguments about dynamic replication or considering the influence of the price of risk explicitly. Third, the parameters of the model are estimated and finally, the option is priced.

This systematic way has been sidestepped in only a few cases, most notably in regard to the volatility parameter of the Black-Scholes [1973] formula. Volatility is the

only parameter in the Black-Scholes formula that is not directly observable. Therefore, practitioners started to find the implied value of volatility by fitting the Black-Scholes model to observed option prices. There were few exchange-traded options for a long time after their launch in the early 1970s, limiting the opportunity for any widespread fitting of parameters other than volatility.

This changed dramatically toward the end of this century. While in the early tests of option pricing such as Rubinstein [1985], the Black-Scholes model was more or less supported, index options in the U.S. can no longer be priced using the simple Black-Scholes formula. Ever since the stock market crash of 1987, the volatility “smile,” i.e., the graph of implied volatility against strike price, for index options tends to slope downward (the smile is rather a “sneer” or a “skew”), while the Black-Scholes model would require a flat volatility smile. Unfortunately, this is only the general pattern. The smile for very low strike prices can flatten out at a high plateau, producing a tilde-shaped pattern. Also, when we observe very high strike prices, the implied volatilities can rise again, exhibiting the traditional u-shaped pattern. Finally, the smile patterns described become more pronounced as the time to expiration shortens, and the smile becomes flatter as the time to expiration lengthens.

Tompkins [1998] finds this same pattern in index options written on the Japanese,

German, and British markets. Other markets also exhibit volatility smiles: Toft and Prucyk [1997] find that individual stock options often exhibit more gently downward-sloping smiles, and Campa, Chang, and Reider [1998] find that foreign exchange options often exhibit u-shaped smiles.

As researchers looked for new option pricing models that were able to explain these option prices, some tried to learn more about the stochastic process of the asset from the observed option prices. This is becoming more possible, since we can now find as many as a hundred simultaneously traded options on the S&P 500 index, all differing in strike price and/or time to expiration. Given such plentiful data, it is a small step to ask what those options tell us about the risk-neutral stochastic process of the underlying security.

Rubinstein [1994], Derman and Kani [1994], and Dupire [1994] have answered this question in a series of articles on a class of models that Rubinstein [1994] terms implied binomial trees. These implied trees are extensions of the original Cox, Ross, and Rubinstein [1979] binomial trees.

I provide a partial survey of this literature. Section I starts with the concept of risk-neutral probabilities, and shows how to back out these probabilities from a set of option prices with a given time to expiration. Section II extends this concept of the risk-neutral probability distribution for a particular time to expiration to cover the whole stochastic process of the asset price across all times. Implied binomial trees and their extensions provide here a feasible framework. The current frontier in research is represented by implied binomial trees that incorporate stochastic volatility.

In section III I discuss other non-parametric option pricing models. Many of these models can be viewed as alternative approximations of the volatility surface, which do not share the implied tree structure of the asset price nodes but rather use (rectangular) grids in time and asset price.

## I. RISK-NEUTRAL PROBABILITIES

State-contingent prices ( $\pi_i$ ) are the prices of securities that promise one dollar if a certain state (i) occurs at a given date, say, T, and zero otherwise. The sum of such state-contingent prices across all date T states has to equal the price of a risk-free bond that pays off one dollar at maturity (T) for sure.

Often it is convenient to normalize the state-con-

tingent prices by the inverse of the price of the bond. The resulting risk-neutral probabilities ( $P_i$ ) then sum to one:

$$P_i = r^T \pi_i \quad (1)$$

where r is  $1 +$  the annual yield of the bond. We can then value any security (C) with payoff ( $X_i$ ) at time T by calculating the expectation under the risk-neutral probabilities:

$$C = E_p[X]/r^T \quad (2)$$

In a complete market, there are either as many securities traded as there are states of the world, or there are a sufficient number of intermediate trading times so that there are as many dynamically rebalanced portfolios feasible as there are states of the world. In such complete markets, Ross [1976] shows that one can recover the complete risk-neutral probability distribution from a set of European option prices. This distribution turns out to be unique, and Breeden and Litzenberger [1978] give an exact formula for it:

$$\frac{\partial^2 C}{\partial K^2} = \pi_{S=K} \quad (3)$$

The second derivative of a European call price (C) taken with respect to its strike price (K) is the state-contingent price ( $\pi$ ) of the future asset price ending up at exactly the strike price of the option. In order to deduce the risk-neutral probabilities from option prices, we can follow any of three approaches. First, find a candidate risk-neutral distribution, and fit the resulting option prices to the observed prices. Second, fit a function of option prices across strike prices through the observed option prices. Then, use the Breeden and Litzenberger [1978] model to obtain the risk-neutral distribution. Third, fit a function of implied volatilities across strike prices through the observed implied volatilities. Next, calculate the function of option prices across strike prices, and, finally, obtain the risk-neutral distribution.

These methods have been applied in numerous forms, and each has some general properties that affect its performance. Fitting the risk-neutral distribution, for example, is problematic since probability distributions need to follow certain shapes: They tail off to zero for small and large stock prices; the center of the distribution is often hump-shaped; and the total probability under the distribution needs to sum to one.

Fitting the function of option prices across strike prices is hardly ever attempted since the function takes the value of the stock price less the present value of the dividend payout over the option's life at a strike price of zero, and approaches a value of zero for very high strike prices. Moreover, the option prices are bounded by no-arbitrage conditions.

Fitting the function of implied volatilities is probably the best method numerically. In a Black-Scholes economy the smile is just a flat line. Also, as long as the smile is not too curved, the corresponding risk-neutral distribution will be well-behaved.

The exact correspondence of risk-neutral distributions to sets of option prices, however, depends critically on complete markets. If markets are incomplete, then there are states of the world for which we cannot replicate the option. This can arise either because of trading restrictions or because there are additional non-traded factors, such as stochastic volatility, or stock price jumps. Then, the observed option prices can be supported by a whole set of risk-neutral distributions and not just by one unique distribution. In turn, this means that we can only establish bounds on option prices, but no longer determine unique option prices across strike prices.

Masson and Perrakis [1997] explore those bounds when there is a jump component present. Bernardo and Ledoit [1999] and Cochrane and Saa-Requejo [1999] present bounds that are based on limits on the profitability of investments. The ratio of gains to losses is restricted in the first article, and the Sharpe ratio (expected return per unit of volatility) is restricted in the latter.

Given the limited availability of option prices, empirical research was limited at the time of the early articles. Banz and Miller [1978] is a notable exception; they show a capital budgeting application of state prices derived from sets of option prices. They do not use actual option prices, however, but effectively end up calculating the required option prices using the Black-Scholes formula.

During the 1980s, there was a lull in activity, and Longstaff [1990] proposes the earliest method in the modern literature. Like Banz and Miller [1978], he derives a histogram of probabilities directly from option prices by assuming that the probabilities are uniformly distributed between any two neighboring option strike prices. Unfortunately, his scheme turns out to be numerically unstable and exhibits negative probabilities unless there are many available options. Mayhew [1995] investigates the stability of this method in detail.

Two survey articles include Cont [1997] and Bahra [1997]. Cont [1997] focuses on particular methods for recovering the risk-neutral probability distributions, while Bahra [1997] divides his attention evenly between methods and applications. Cont [1997] also provides the basis for a framework for classifying the different methods into expansion, non-parametric, and parametric methods. We use related but slightly different categories, given that the two basic extensions of the lognormal distribution are to either find a more flexible parametric distribution or to try a non-parametric method.

### Parametric Methods

Parametric methods can be divided into three groups. First, expansion methods start with a basic distribution, and add correction terms in order to make it more flexible. Second, generalized distribution methods introduce more flexible distributions with additional parameters beyond the two parameters of the normal or lognormal distributions. Third, mixture methods create new distributions from mixtures of simple distributions.

Expansion methods are conceptually related to the expansion of a function in a Taylor series around a point. They typically start with a simple (often normal or lognormal) distribution, and then add correction terms in order to obtain a more flexible shape of the risk-neutral distribution, which is then fitted to the observed option prices. A general concern in this class of methods is that for certain parameter constellations the risk-neutral distributions can take on negative values. Thus, all results have to be checked for violations of the positivity constraint.

Abadir and Rockinger [1997] use confluent hypergeometric (or Kummer's) functions as a basis for the risk-neutral density and derive the option price function across strike prices in closed-form. Abken, Madan, and Ramamurtie [1996a, 1996b] use four-parameter Hermite polynomials. These polynomials are related to successive derivatives of the normal density function and provide four correction terms to the normal distribution.

Using the same polynomials but with restrictions on some of the coefficients, Jondeau and Rockinger [1998a] and Corrado and Su [1997] restrict their attention to the class of Gram-Charlier expansions. Jondeau and Rockinger [1998a] also provide an algorithm that forces the coefficients, which are based on the moments of the risk-neutral distribution, to lie within the region that guarantees positive probabilities.

Brenner and Eom [1997] use Laguerre polynomials, which are corrections to a gamma distribution. As a main benefit, the authors note that the approximated risk-neutral distribution has unbiased moments.

Jarrow and Rudd [1982], Corrado and Su [1996], Longstaff [1995], and Rubinstein [1998] use Edgeworth expansions. Here, the base distribution (lognormal, lognormal, normal, and binomial, respectively) is augmented with correction terms that successively match each of the first four cumulants. These cumulants are closely related to the moments of the option-implied distribution. The authors then search for the set of cumulants that gives the best fit between observed option prices and option prices based on the expanded risk-neutral distribution.

Potters, Cont, and Bouchaud [1998] use cumulant expansions to add a single correction term to a normal distribution, which adjusts for the kurtosis of the risk-neutral distribution. This is a somewhat atypical choice since most expansions also try to correct at least for skewness, that is, for the downward-sloping pattern in the implied volatilities. They are able to express their correction as an additional quadratic term in the function of implied volatilities across strike prices.

Generalized distribution methods use distributions with more inherent flexibility, which tend to have one or two additional parameters beyond the two parameters of the normal and lognormal distributions. Generalized distributions often include different common distributions as limiting cases for particular parameter constellations.

Aparicio and Hodges [1998] use generalized beta functions of the second kind. This family of four-parameter distributions was first explored by Bookstaber and McDonald [1987] and includes the lognormal, gamma, and exponential distributions, several Burr type distributions, and others. Posner and Milevsky [1998] use the Johnson family of densities. Johnson [1949] proposes translating normal distributions through logarithmic and hyperbolic sine transformations while adding two additional parameters.

Sherrick, Garcia, and Tirupattur [1995] use the Burr III distribution on a data set of options on soybean futures. Sherrick, Irwin, and Forster [1992 and 1996] use the Burr XII distribution on S&P 500 futures options. Curiously, none of these three articles mentions how they accommodate the early exercise that is possible for American options on futures. It would be incorrect to just ignore the possibility of early exercise and proceed as if the options were European options.

Mixture methods achieve greater flexibility by drawing with different probabilities from several simple distributions, each with a distinct parameterization. A problem with mixtures is that the number of parameters increases very fast. For example, mixing three lognormal distributions yields eight parameters. Two parameters are used for each lognormal and two mixing probabilities are added. The third mixing probability is determined by the requirement that the mixing probabilities have to sum to one.

Given that there are often only ten or twenty observed option prices across different strike prices, mixture methods can easily overfit the data. Still, mixtures tend to be somewhat more flexible and are capable of generating a wider variety of shapes for the probability distributions than generalized distributions.

Melick and Thomas [1997] use mixtures of three lognormal distributions. They apply their method to American options on crude oil futures, and appropriately bound the American option prices to account for the possibility of early exercise. Ritchey [1990] gives examples of option prices where the risk-neutral distribution of the log returns is a mixture of normal distributions.

### **Non-Parametric Methods**

Non-parametric methods try to achieve greater flexibility in fitting the risk-neutral distribution to option prices, too. Rather than requiring a parametric form of the distribution, they allow more general functions. Again, we divide the methods into three groups.

First, kernel methods are conceptually related to regressions in that they try to fit a function to observed data. The main difference is that kernel regression does not specify the parametric form of the function.

Second, maximum-entropy methods find a non-parametric probability distribution, which is as close as possible in terms of information content to a prior distribution while satisfying certain constraints such as pricing observed options correctly.

Third, curve fitting methods are a loosely associated group of methods that try to fit the implied volatilities or the risk-neutral probability as well as possible with some flexible function. A commonly used measure of fit is the sum of squared differences between observed option prices and option prices suggested by the curve-fitting method.

Kernel methods are based on the concept that each data point (say, the implied volatility at a certain

strike price) should be viewed as the center of a region where the true function could be lying. The farther away a point is from the observed data point, the less likely it is that the true function goes through that distant point. A kernel  $k(x)$ , which is often assumed to be the normal distribution  $n(x) = (1/\sqrt{2\pi})e^{-0.5x^2}$ , measures the corresponding drop in likelihood when we move away from the data point.

Thus, if we observe implied volatilities across strike prices  $\sigma_i(K_i)$ , the kernel regression in one dimension is:

$$\sigma(K) = \frac{\sum_{i=1}^n k\left(\frac{K - K_i}{h}\right) \sigma_i}{\sum_{i=1}^n k\left(\frac{K - K_i}{h}\right)} \quad (4)$$

where  $h$  is the bandwidth, which governs the smoothness of the kernel regression. Kernel regressions tend to be very data-intensive, which limits their use. Also, they tend to fill in missing data in a non-intuitive way. Thus, in a downward-sloping implied volatility smile where ranges of strike prices have no observations, we find question-mark shaped segments filling those gaps, rather than smooth interpolations across the gaps.

Ait-Sahalia and Lo [1998a] use kernel estimators across the five dimensions of stock price, strike price, time to expiration, interest rate, and dividend yield. They also use a reduced set of three dimensions, which are forward price, strike price, and time to expiration.

Pritsker [1997] employs their methodology in order to investigate the risk-neutral distribution of interest rates. He warns, however, that the presence of persistence in interest rates violates the assumption of independently and identically distributed observations, and thus the asymptotic results of Ait-Sahalia and Lo [1998a] do not need to hold. Indeed, Pritsker [1997] finds severe finite sample biases.

In a related study, Rookley [1997] suggests a bivariate kernel estimator across time to expiration and moneyness. In addition, he proposes to model the stochastic process of implied intraday volatilities.

Maximum-entropy methods effectively try to maximize the amount of missing information, given the constraints; that is, they try to be the least prejudiced with respect to unknown or missing information. In order to implement these ideas — and somewhat similar to a Bayesian framework — a prior distribution ( $O_i$ ) is chosen,

and the posterior risk-neutral distribution ( $P_i$ ) is found by maximizing the cross-entropy:

$$-\sum_i P_i \ln\left(\frac{P_i}{O_i}\right) \quad (5)$$

subject to constraints such as positivity of the probabilities, summing of the probabilities to one, and correct pricing of options and the underlying. If the prior distribution happens to be the uniform distribution, then the cross-entropy reduces to the simple entropy:

$$-\sum_i P_i \ln P_i$$

Buchen and Kelly [1996] use the uniform and the lognormal distribution as prior distributions and show some simulation results. Stutzer [1996] derives the risk-neutral distribution using the historical distribution of the asset price as a prior. He constrains the risk-neutral distribution only to price the underlying asset correctly. Then, he prices the options and finds evidence of a volatility skew (which depends rather delicately on the length of the historical sample used to derive the historical distribution).

Curve-fitting methods are a broad class of methods in which either the function of implied volatilities across strike prices or the risk-neutral distribution itself is approximated by some general function. The functions involved tend to be flexible and smooth such as the class of polynomials in the strike price.

For added flexibility, such polynomials can be pieced together in a smooth fashion into splines. This is achieved by connecting the polynomial segments smoothly at so-called knots by matching the levels and some derivatives of the segments at the knots. The main advantage of splines is that the knots serve to localize the fit of the polynomials and thereby avoid oscillating and extreme values of the polynomials in regions with little data.

Finally, smoothness can be made an explicit objective in non-parametric curve-fitting methods by finding a function through the data that fits the data in a least squares sense and also minimizes the integral of squared second derivatives, which accounts for the cumulative curvature of the function.

We investigate the methods fitting the smile first, and then the methods fitting the risk-neutral distribution. Shimko [1993] introduced the first method of curve-fit-

ting to the smile when he fitted a quadratic polynomial. He then translates the implied volatilities into option prices and can derive the risk-neutral distribution from the second derivatives of the option prices with respect to strike prices according to Breeden and Litzenberger [1978]. He finally grafts lognormal tails onto the risk-neutral distribution.

Malz [1997] proposes a variation of this method in which a quadratic polynomial is fitted to the implied volatilities across call option deltas instead of strike prices. The deltas are the derivatives, according to the Black-Scholes formula, of the option prices with respect to the stock price. The benefit is avoiding grafting tails onto the distribution. Shimko's [1993] approach is extended in Brown and Toft [1999] to seventh-order splines.

Campa, Chang, and Reider [1998] use cubic splines to fit the volatility smile. Aparicio and Hodges [1998] use cubic B-splines that belong to a particular family of polynomials used for the segments.

Rosenberg and Engle [1997] use a polynomial fitted to the log of the smile, which then automatically prevents negative implied volatilities. Rosenberg [1996] extends this method to the bivariate case. Jackwerth [1999] maximizes the smoothness of the smile and can then control the trade-off between option price fit and smoothness explicitly.

The calculations in curve-fitting the smile do not require iterations and can be computed in one pass and closed form. They do not guarantee, however, that the resulting probabilities will always be positive, a condition that has to be checked separately.

The second group of curve-fitting methods tries to approximate the risk-neutral distribution directly. Rubinstein [1994] minimizes the distance of discretized probabilities to a lognormal prior distribution subject to correctly pricing the observed options and the underlying. Jackwerth and Rubinstein [1996] show that the different ways to measure this distance, such as by computing squared or absolute deviations or cross-entropy, do not affect the resulting risk-neutral distribution by much as long as we observe more than a few option prices. Jackwerth and Rubinstein [1996] also introduce a new method that does not need a prior distribution. They maximize the smoothness of the risk-neutral distribution subject to correctly pricing the options.

Mayhew [1995] uses the same idea of maximizing the smoothness of the distribution and approximates the density with cubic splines. Hartvig, Jensen, and Pedersen [1999] suggest a related method in which they build up

the logarithm of the risk-neutral distribution from piecewise linear segments. In a Bayesian framework, they then use an exponential prior distribution and find the risk-neutral distribution that accords with the observed options prices.

Although there are numerous methods for recovering the risk-neutral distributions, the results tend to be rather similar unless we have only very few option prices. The reason is that if we observe a set of as few as ten to fifteen options, the collective set of no-arbitrage bounds is so tight that the price of an additional option that has not been priced yet is often determined to within the bid/ask spread. This holds less in the tails of the distribution but is a rather accurate assessment of the situation for near-the-money options. Jackwerth and Rubinstein [1996] make this point that one can use just about any reasonable method and the resulting risk-neutral distributions will be rather similar.

Campa, Chang, and Reider [1998] corroborate this finding when they use three alternative methods — cubic splines, Rubinstein's [1994] minimal-distance method, and mixtures of lognormal distributions — to fit the volatility smile of options on exchange rates. Again with similar results, Coutant, Jondeau, and Rockinger [1998] use three alternative methods — a mixture of lognormal distributions, Hermite polynomial expansion, and maximum-entropy — to fit options on bond futures.

As a practical matter, approximating the smile tends to be easier than directly approximating the probability distribution. In the Black-Scholes reference case, the smile is just a flat line as opposed to the lognormal distribution, which is highly non-linear. This advantage carries over to the actual, skewed volatility smiles, too.

## Applications

One straightforward application of the risk-neutral density is to price any payoff with the same time to expiration. Thus, if we have a set of liquid options from which to derive the risk-neutral distribution, we can then use it to price illiquid options. These options can have exotic payoffs, but they must be European, since the risk-neutral distributions cannot handle early exercise without additional assumptions.

A number of authors have investigated the shape of the risk-neutral probability distribution around announcements of economic news. Campa and Chang [1996 and 1998] and Campa, Chang, and Refalo [1999] use cross-rate foreign exchange options from the

exchange rate mechanism (ERM) and the Brazilian real/U.S. dollar, respectively, in order to assess the credibility of commitments to exchange rate target zones. They assess the minimum and maximum risk-neutral probability associated with the exchange rate leaving the target band. These articles show that option market probability assessments often rise prior to exchange rate crises and realignments.

Mizrach [1996] finds evidence in option prices that the market had incorporated the potential withdrawal of the pound from the ERM in advance of the actual announcement. For each day he fits a mixture of three lognormal distributions to option prices. His finding is that the option markets seem to reflect an increased risk-neutral probability assessment of a devaluation a few days prior to the actual devaluation.

McCauley and Melick [1996] and Malz [1997] investigate general market sentiments in foreign exchange markets by quite literally interpreting graphs of the change in the shape of the risk-neutral distribution for different times. Bahra [1997] argues similarly when investigating the change in shape of the risk-neutral distribution around announcements of an inflation report and of the German money supply (M3). He then goes on to provide a general discussion of interpreting graphs of risk-neutral probability distributions in assessing monetary conditions and credibility, identifying market anomalies, and investigating the timing and effectiveness of monetary operations. Finally, Brenner, Eom, and Landskroner [1996] imply the foreign exchange rate from option prices but they do not use the whole risk-neutral distribution.

Söderlind and Svensson [1996] fit mixtures of lognormal distributions to options on bonds in order to recover risk-neutral distributions of interest rates and provide graphs of the distributions on two dates. Coutant, Jondeau, and Rockinger [1998] investigate the daily implied interest rate distribution and focus on the time around the French 1997 snap elections. Jondeau and Rockinger [1998b] analyze the same elections using the options on the franc/deutschemark exchange rate. Both these latter articles use a number of different methods, among them mixtures of lognormals and Hermite polynomial expansions, which all yield similar results. They conclude that the election results were anticipated by the market before the actual announcement, since the volatility of the risk-neutral distributions increased.

Melick and Thomas [1997] use the crude oil futures option market in order to assess the impact of the

Gulf War on the risk-neutral distributions. They fit a mixture of three lognormal distributions to daily option prices and plot the implied risk-neutral probabilities of large price moves relative to the current future price. Using the same method, Leahy and Thomas [1996] investigate the impact of the 1995 Quebec sovereignty referendum on the Canadian foreign exchange market.

As a general observation, there is a danger in reading too much information into the graphs of risk-neutral distributions. After all, they are not graphs of actual distributions but mental constructs that are used for pricing purposes. Also, none of the empirical articles investigating the change in shape of the risk-neutral distribution provides a formal argument of when the shapes should be considered different. So far, they tend to appeal to some intuitive notion along the lines of “tails are looking fatter after the event” or “a certain moment increased over time.” Such judgments do not account for estimation error due to observation of only noisy option prices.

In particular, none of these researchers provide for a statistical test of the significance or controls for exogenous factors that could have caused the change in shape. For example, increased volatility could have been driven by a marketwide rise in volatility rather than by the particular economic announcement.

A rather different line of research has looked at the relationship between probabilities and utility functions in a representative agent setting. There, marginal utility ( $U'$ ) is closely related to the ratio of risk-neutral to subjective probabilities ( $Q$ ), where the historical distribution is a proxy for the subjective probabilities:

$$U' = \frac{\lambda P}{r^T Q} \tag{6}$$

where  $\lambda$  is the shadow price of the budget constraint.

Rosenberg and Engle [1997] and Ait-Sahalia and Lo [1998b] derive such marginal utility functions. Jackwerth [1999] goes on to recover implied risk aversion functions:

$$RA = -\frac{U''}{U'} = \frac{Q'}{Q} - \frac{P'}{P} \tag{7}$$

The risk aversion functions turn out to be well-behaved pre-crash, and seem to imply mispricing in the option market post-crash. Stutzer [1996] uses maximum-entropy



arguments to find the risk-neutral distribution that is most compatible with the historical distribution subject to having the mean equal to the risk-free rate.

As a general observation, we find that of the three entities (risk-neutral distribution, subjective distribution, and utility function), we can derive one as long as we know the other two. Moreover, all involved quantities are difficult to estimate. Risk-neutral distributions are rather well-understood, and subjective distributions can be derived from historical distributions with some assumptions on the updating of beliefs by investors. Utility functions are the least understood and are notoriously hard to estimate.

## II. IMPLIED BINOMIAL TREES

Once we derive a risk-neutral distribution of the terminal asset price, it is tempting to ask how the asset price can reach that terminal distribution. In other words, we try to model a stochastic process for the asset price that results in the same terminal distribution that is implied in the option prices. Unfortunately, there is no such unique stochastic process. Moreover, even if we knew the risk-neutral distribution for every time between today and the longest-dated time to expiration, there would still not be a unique stochastic process. The reason is that the stock price could evolve according to either a diffusion process, a jump process, or any number of alternative processes, which can involve jumps or additional stochastic factors such as stochastic interest rates or stochastic volatility.

If we are willing to limit ourselves to the class of generalized one-dimensional diffusion processes, and if we are willing to rely solely on observed option prices in the market, implied binomial trees are a potential solution. We can view implied binomial trees as discretizations of generalized one-dimensional diffusions in which the generalized volatility parameter is allowed to be a function of time and the asset price. This extends the standard diffusion of the Black-Scholes formula whose volatility parameter is constant across time and asset price. The risk-neutral drift in either the standard or the generalized diffusion processes is fully determined by the risk-free rate. Discretizing the volatility surface as a binomial tree is then one convenient model choice, but not the only one.

### One-Factor Implied Trees

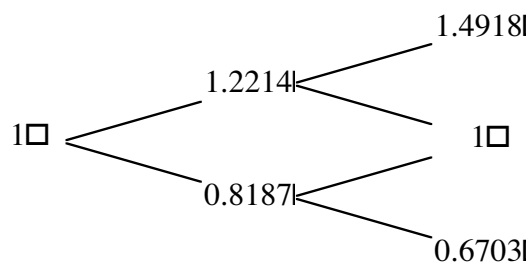
We start out with a simple two-step example within a standard Cox, Ross, and Rubinstein [1979]

binomial tree. This binomial tree is the discretization of the continuous-time Black-Scholes model. We assume that the volatility is 0.2, the current stock price ( $s_t$ ) is 1, and the interest return ( $r$ ) is 1.1. Also, we assume that there are no dividends being paid and that each time step spans one year. Exhibit 1 shows the standard binomial tree for the stock price, and Exhibit 2 displays the probabilities associated with the different nodes.

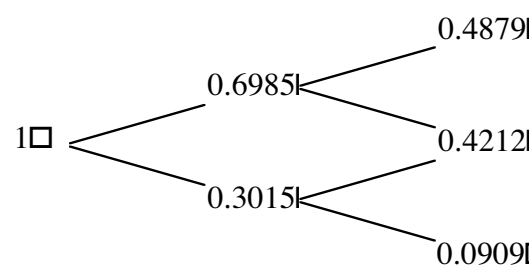
Let us now complicate our simple world by introducing a volatility smile. In particular, for every 10 percentage points of moneyness subtracted (added) from the at-the-money option, volatility increases (decreases) by a factor of 1.1. Thus, a call option with strike price 1.1 has an implied volatility of  $0.2/1.1 = 0.1818$  and, with two years until expiration, a price of 0.1497. Then, Rubinstein [1994] suggests building an implied binomial tree backward from the terminal distribution to the starting date.

We take the terminal stock prices from the standard binomial tree. The implied risk-neutral probabilities at the terminal nodes of the implied binomial tree have to price the bond, the stock, and the option, respectively:

**EXHIBIT 1**  
Stock Prices for the Standard Binomial Tree



**EXHIBIT 2**  
Probabilities for the Standard Binomial Tree





$$0.8264 = (1p_1 + 1p_2 + 1p_3)/1.1^2$$

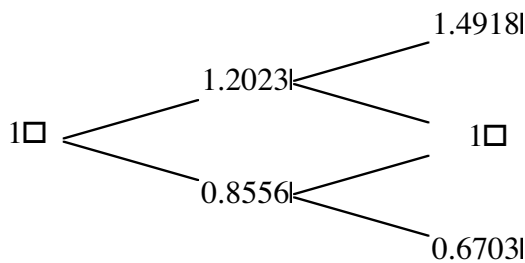
$$1 = (0.6703p_1 + 1p_2 + 1.4918p_3)/1.1^2$$

$$0.1497 = (0p_1 + 0p_2 + 0.3918p_3)/1.1^2$$

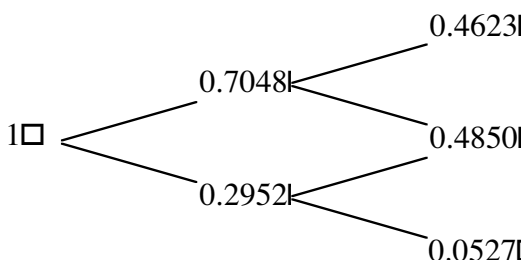
Solving the system of equations indicates that the resulting probabilities are  $p_1 = 0.0527$ ,  $p_2 = 0.4850$ , and  $p_3 = 0.4623$ . A key assumption is that the path probabilities for all paths reaching the same terminal asset price are equal. Then, a unique implied binomial tree can be derived by backward recursion that is guaranteed to be arbitrage-free.

The nodal probability at the terminal center node of 0.4850 is evenly split up and ends up at the two nodes of the intermediate step. Thus, the upper intermediate node receives a total probability of 0.7048, which derives from a probability of 0.4623 of reaching the next-higher node and of 0.2425 ( $= 0.4850/2$ ) of reaching the next-lower node. The probability at the lower intermediate node is calculated similarly.

### EXHIBIT 3 Stock Prices for the Implied Binomial Tree



### EXHIBIT 4 Probabilities for the Implied Binomial Tree



Finally, we can solve for the upper-intermediate stock price. First we need the transition probability of moving up, which is  $0.4623/0.7048 = 0.6559$ , and of moving down, which is  $0.2425/0.7048 = 0.3441$ . The stock price is then  $(0.6559 \times 1.4918 + 0.3441 \times 1)/1.1 = 1.2023$ . The stock price at the lower intermediate node is calculated similarly. The resulting trees are shown in Exhibits 3 and 4.

Rubinstein's [1994] approach has two drawbacks. First, prices of options expiring earlier than the terminal date are not used in constructing the tree. Second, the assumption of equal path probabilities is rather arbitrary and restrictive.

Jackwerth [1997] addresses these concerns by generalizing the implied binomial tree through the introduction of a simple weight function. This weight function governs the backward construction of the implied binomial tree. Rubinstein's [1994] tree corresponds to a special, linear weight function. With this generalization, we are able to incorporate information from times other than the end of the tree and also from American and exotic options.

A call option with strike price 1.1 and time to expiration of one year is priced in the market at 0.0724. The option is not correctly priced on the implied binomial tree, however, where it would be valued at  $(0.7048 \times 0.1023 + 0.2952 \times 0)/1.1 = 0.0656$ . We achieve the appropriate pricing by changing the split according to which the probability at the terminal center node is distributed backward.

The implied binomial tree splits the probability evenly, while the — properly fitted — generalized binomial tree sends  $0.4190 \times 0.4850 = 0.2032$  to the intermediate upper node and  $0.5810 \times 0.4850 = 0.2818$  to the intermediate lower node. Thus the upper intermediate node receives a total probability of  $0.6655 = 0.4623 + 0.2032$ . The probability at the lower intermediate node is calculated similarly.

Finally, we can solve for the intermediate stock prices as above. The resulting generalized binomial shown in Exhibits 5 and 6 accurately prices the call option at 0.0724.

Derman and Kani [1994] introduce an alternative implied tree. It is constructed forward by adding the nodes for a new time step based on the information in a particular set of option prices. These options have strike prices equal to the asset prices of the previous step, and they expire at the next time step. Thus, options with different maturities can be incorporated; but in order to

obtain the required set of option prices, extensive interpolation and extrapolation using the observed option prices is needed.

Also, the tree is not necessarily arbitrage-free, since negative probabilities can occur. These negative probabilities have to be reset to values between zero and one in an ad hoc fashion. As a result, Derman and Kani [1994] trees become numerically unstable, especially when the number of steps is large.

For an improvement, Derman, Kani, and Chriss [1996] suggest trinomial trees in which additional degrees of freedom are used in order to ameliorate the numerical problems. Chriss [1996] shows how to incorporate American options, but he needs a costly optimization step at each node of the implied tree.

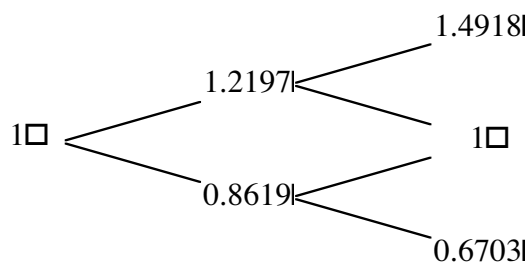
Barle and Cakici [1998] add some other improvements to increase the stability of the original Derman and Kani [1994] trees. For one, they suggest aligning the center nodes of the tree with the forward price rather than with the current stock price. Also, whenever a new option is needed to provide informa-

tion about the next node of the tree, they advocate setting the strike price equal to the forward value of the previous node's stock price instead of using the previous node's stock price itself as originally suggested in Derman and Kani [1994]. Finally, Barle and Cakici [1998] recommend using the Black-Scholes formula in order to price the needed options as opposed to pricing them on a binomial tree.

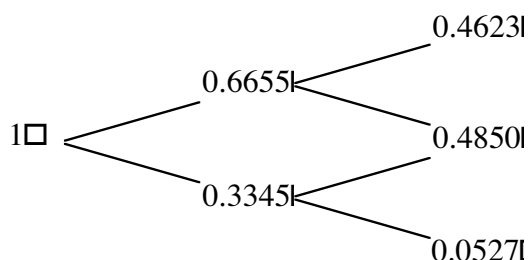
As shown in Exhibits 7 and 8, the implied binomial tree by Derman and Kani [1994], with the corrections suggested in Barle and Cakici [1998], requires the calculations as follows. We follow closely the notation of Barle and Cakici [1998]. The lower intermediate stock price follows from ( $i = 1$ ):

$$S_i = F_i \frac{\lambda_i F_i - \Delta_i^C}{\lambda_i F_i + \Delta_i^C} = 1.1 \frac{1(1.1) - 0.0797}{1(1.1) + 0.0797} = 0.9514 \quad (8)$$

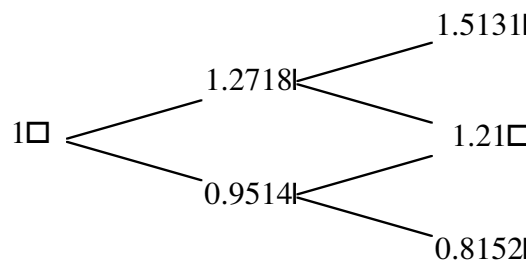
**EXHIBIT 5**  
Stock Prices for the Generalized Binomial Tree



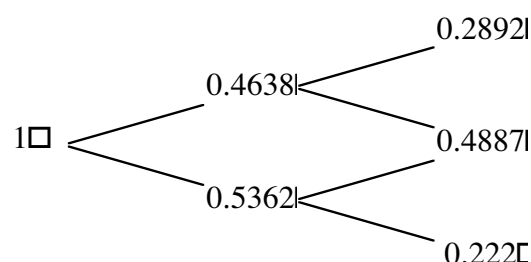
**EXHIBIT 6**  
Probabilities for the Generalized Binomial Tree



**EXHIBIT 7**  
Stock Prices for the Derman and Kani Tree  
with Corrections by Barle and Cakici



**EXHIBIT 8**  
Probabilities for the Derman and Kani Tree  
with Corrections by Barle and Cakici



where  $F_i = S_i(r/d)^{\Delta t} = 1.1$ ;  $S_i$  is today's stock price;  $\Delta t$  is one time step of one year;  $\lambda_i$  is the state-contingent price today ( $= 1$ );  $\Delta_i^C = r^{\Delta t}C(F_i, t_{n+1}) - \sum_{j=i+1}^n \lambda_j(F_j - F_i)$ ; and  $C(F_i, t_{n+1}) = 0.0724$  is the call expiring next period with strike price  $F_i$ . The upper intermediate stock price follows from  $S_{i+1} = F_i^2/S_i = 1.21/0.9514 = 1.2718$ . The upper intermediate probability can then be derived from:

$$p_i = \frac{F_i - S_i}{S_{i+1} - S_i} = \frac{1.1 - 0.9514}{1.2718 - 0.9514} = 0.4638 \quad (9)$$

and the lower intermediate probability is  $1 - 0.4638 = 0.5362$ .

In order to construct the terminal step, we start with the center node

$$S_{(n/2)+1} = S_1 \left( \frac{r}{d} \right)^{t_{n+1}} = 1.1^2 = 1.21$$

where  $n = 2$ . Next we find the upper terminal node ( $i = 2$ ):

$$\begin{aligned} S_{i+1} &= \frac{\Delta_i^C S_i - \lambda_i F_i (F_i - S_i)}{\Delta_i^C - \lambda_i (F_i - S_i)} \\ &= \frac{(0.03)1.21 - (0.4216)1.399(1.399 - 1.21)}{0.03 - 0.4216(1.399 - 1.21)} \\ &= 1.5131 \end{aligned} \quad (10)$$

where  $F_i = S_i(r/d)^{\Delta t} = 1.2718(1.1) = 1.399$ ;  $S_i$  is the stock price at the upper intermediate node;  $\lambda_i = p_i/r^{\Delta t} = 0.4638/1.1 = 0.4216$  is the state-contingent price at the upper intermediate node; and  $C(F_i, t_{n+1}) = 0.03$  is the call expiring in the last period with strike price  $F_i$ .

Finally, we find the lower terminal node ( $i = 1$ ):

$$\begin{aligned} S_i &= \frac{\lambda_i F_i (S_{i+1} - F_i) - \Delta_i^P S_{i+1}}{\lambda_i (S_{i+1} - F_i) - \Delta_i^P} \\ &= \frac{(0.4875)1.0465(1.21 - 1.0465) - (0.0467)1.21}{0.4875(1.21 - 1.0465) - 0.0467} \\ &= 0.8152 \end{aligned} \quad (11)$$

where  $F_i = S_i(r/d)^{\Delta t} = 0.9514(1.1) = 1.0465$ ;  $S_i$  is the stock price at the lower intermediate node;  $\lambda_i = p_i/r^{\Delta t} =$

$0.5362/1.1 = 0.4875$  is the state-contingent price at the lower intermediate node;  $\Delta_i^P = r^{\Delta t}P(F_i, t_{n+1}) - \sum_{j=i}^{i-1} \lambda_j(F_j - F_i)$ , and  $P(F_i, t_{n+1}) = 0.0467$  is the put expiring in the last period with strike price  $F_i$ .

As a last step, we calculate the transition probability between the upper intermediate node and the upper terminal node:

$$p_i = \frac{F_i - S_i}{S_{i+1} - S_i} = \frac{1.399 - 1.21}{1.5131 - 1.21} = 0.6236 \quad (12)$$

and similarly the transition probability between the lower intermediate node and the center terminal node:

$$p_i = \frac{F_i - S_i}{S_{i+1} - S_i} = \frac{1.0465 - 0.8152}{1.21 - 0.8152} = 0.5859 \quad (13)$$

We can then calculate the probabilities of reaching the terminal nodes. The lower node receives  $0.5362(1 - 0.5859) = 0.222$ ; the center node receives  $0.4638(1 - 0.6236) + 0.5362(0.5859) = 0.4887$ ; and the upper node receives  $(0.4638)0.6236 = 0.2892$ .

Dupire [1994] suggests a trinomial tree similar to Derman and Kani [1994], which is also forward-constructed, and the nodes within each additional step are determined by the prices of options expiring on that time step. Computationally, the generalized binomial tree by Jackwerth [1997] is a good choice among the backward-constructed trees. It offers the flexibility of incorporating information on exotic and American options, and it is guaranteed to be arbitrage-free. The computations are simple, but the calibration can involve a non-linear optimization in order to fit the tree to non-European options.

Alternatively, the tree by Barle and Cakici [1998] is a good choice among the forward-constructed trees, although it can handle only European options and can still exhibit arbitrage violations (but much less frequently than the Derman and Kani [1994] trees). The calculations are slightly more convoluted, and the implied volatilities of observed options have to be interpolated and extrapolated in order to price the options required to construct the next node of the tree.

## Two-Factor Implied Trees

Paralleling the development of parametric option pricing models, implied binomial tree models have been

extended to multiple factors. So far in the literature, there are three implied tree models, which can incorporate stochastic volatility as well as the usual process of the asset price.

Derman and Kani [1998] start out by fitting a trinomial implied tree to the observed option prices. The transition probabilities from node to node are fixed at this stage. In order to incorporate stochastic volatility, they allow the transition probabilities to change according to a separate stochastic process, which has to be specified exogenously. Conceptually, their new tree is related to the interest rate model of Heath, Jarrow, and Morton [1992], and suffers from similar drawbacks; the resulting process is generally non-Markovian since the evolution of the volatility surface is path-dependent. Thus, option prices have to be computed through an expensive Monte Carlo simulation.

Similar in spirit but computationally simpler is the approach of Ledoit and Santa-Clara [1998], who model stochastic implied volatilities as opposed to the stochastic local volatilities in Derman and Kani [1998]. They can then derive the conditions under which the joint process of stock price and volatility surface is arbitrage-free. This joint process can be calibrated to observed option prices. Illiquid or exotic options can then be priced via Monte Carlo simulation.

In a related article, Britten-Jones and Neuberger [1999] are able to describe the set of continuous stochastic processes that is compatible with the observed option prices. In particular, these processes share the same volatility forecasts. Similar to Derman and Kani [1998], they also use implied trinomial trees, but they restrict the stochastic volatility component to be a Markov chain by discretizing volatility into a set of volatility states. Transition probabilities are assigned to move from one volatility state into the next. Finally, they adjust the transition probabilities so that the observed option prices are matched. While losing some of the generality of Derman and Kani [1998], this tree is faster to compute, and no simulations are needed in order to price options.

Neither the Derman and Kani [1998] nor the Britten-Jones-Neuberger [1999] stochastic volatility implied trees has yet been empirically tested.

### **Empirical Tests**

Dumas, Fleming, and Whaley [1998] develop deterministic volatility functions, which are polynomials in time and asset price, and use these functions as volatil-

ity surfaces of generalized diffusions. They alternatively interpret these functions as implied volatility surfaces of the options. The latter ad hoc model performs slightly better, so we describe its estimation in more detail.

The authors collect option prices on the S&P 500 index with time to expiration from six to one hundred days. On every Wednesday from June 1998 through December 1993, they fit the volatility surface to the observed options on that day. They then wait for a week, and use the calibrated surface to forecast the option prices on the following Wednesday, conditioned on the passage of one week's time and the observed movement in the asset price. The average root mean squared prediction error ranges from 47 to 59 cents for the options with time to expiration of more than seventy days. The authors compare this prediction error to an average bid/ask spread of 47 cents.

At this point, one should not throw the baby out with the bathwater and dismiss implied binomial trees altogether; competing fully parametric models do not necessarily perform much better. Bakshi, Cao, and Chen [1997] fit parametric stochastic jump, stochastic volatility, models to S&P 500 index options from June 1988 through May 1991. They predict one-day ahead option prices and report average errors for options with time to expiration from 60 to 180 days. These errors are about 40 cents even for their best-performing models.

While I do not intend to cover the literature on fitting generalized stochastic processes to option prices, I must mention a few articles. Malz [1996] uses option prices to estimate a jump-diffusion and finds that the option-implied realignment probabilities within the European monetary system increased in advance of the actual realignment. A wealth of information on fitting stochastic jump and stochastic volatility processes to option prices is in Bates [1991, 1996a, 1996b, 1998].

Also, Jackwerth and Rubinstein [1998] compare implied binomial trees, parametric models (among them a stochastic volatility model), and naive trader rules of thumb. These naive rules assume that implied volatility is a function of strike price (the absolute smile rule) or moneyness (the relative smile rule), and does not change as time passes. The authors fit all their models to daily observations of options on the S&P 500 index from April 1986 through December 1995. They then calculate the average prediction errors for ten-day (two-week) forecasts of options with times to expiration from 135 to 215 days.

The errors range from 75 cents for the implied binomial tree to 57 cents for the naive trader rules and to

about 78 cents for the parametric models. Given that the empirical standard deviation of these errors is 80 to 130 cents, we can conclude that none of the models performs extraordinarily well and that implied binomial trees perform about as well as the parametric models and the naive trader rules.

### III. OTHER NON-PARAMETRIC MODELS

Implied binomial trees discretize the volatility surface of a generalized one-dimensional diffusion in a particularly convenient way, but there are other choices. An alternative is developed in Dumas, Fleming, and Whaley [1998] under the name of deterministic volatility models. Here, the volatility surface is discretized onto a prespecified (rectangular) grid in time and asset prices. Finite-difference methods are then used to price the options based on the volatility surface.

Andersen [1995] uses the same methodology and highlights different choices for the finite-difference method, such as implicit versus semi-implicit. He also points out the computational advantage of deterministic volatility models over implied binomial trees when pricing exotic options with discontinuous payoffs, such as barrier options.

Again using a deterministic volatility model and finite-difference pricing of options, Bodurtha and Jermakyan [1996a, 1996b] and Lagnado and Osher [1997] require further that the volatility surface be smooth. Both sets of authors acknowledge that their respective methods are rather expensive computationally.

Coleman, Li, and Verma [1998] also try to impose some smoothness requirement on the volatility surface. They employ bicubic splines for spanning the volatility surface, which reduces the computational requirements. For pricing purposes, they use a finite-difference approach. Similar in spirit is the approach of Avellaneda et al. [1996], who use a cross-entropy maximization to fit the volatility surface to be close to some prior surface and to fit the observed option prices. Their approach, however, suffers from numerical difficulties as the volatility surface is riddled with spikes at the data points. Bouchouev and Isakov [1997] build up the volatility surface from rather complex functions that involve integral evaluations. They assume that volatility is independent of time and can then find formulas for the volatility surface. The independence assumption, however, somewhat defeats the purpose of having a deterministic volatility model to begin with. Their approach then effectively

collapses to a very involved method of finding the risk-neutral distribution.

Carr and Madan [1998] also work within the framework of a generalized one-dimensional diffusion process. They add structure to the volatility surface by assuming that the asset price itself is path-independent; that is, it depends only on time and the level of the driving Brownian motion. They can then determine the whole volatility surface by fitting it to one set of observed option prices with common times to expiration.

In the case of implied binomial trees, the model is fitted to the observed option prices, but there is still some structure in the model. For example, the asset price can only move up or down over the next instant of time. Laurent and Leisen [1998] abandon this framework by using a prespecified grid of times and asset prices, but they model the transition probabilities as a Markov chain. Since the asset price can now move to more nodes than just the next-higher and next-lower one, the market is no longer complete. Then, the risk-neutral distribution is not uniquely identified, and there are two choices. One has to either pick one of the distributions or only establish bounds around the observed option prices. The authors suggest either picking the distribution that yields the highest prices for all non-traded assets, or using cross-entropy maximization to find a distribution that is close to a prior distribution and also fits observed option prices.

In related work, Samperi [1996] investigates incomplete markets and gives expressions for the bounds on option prices if the volatility surface is bounded by a highest and lowest value. He then assumes that investors follow risk-minimizing trading strategies and is able to derive a unique implied tree as a solution.

Beyond these models lie the fully non-parametric models such as kernel methods, which discard even the little remaining structure of generalized diffusion processes and view the option pricing model as a non-linear surface of option prices or implied volatilities that is fitted to the observed prices. For example, Ait-Sahalia and Lo [1998a] fit a five-dimensional surface of option-implied volatilities across stock price, strike price, time to expiration, interest rate, and dividend yield.

Similarly, Hutchinson, Lo, and Poggio [1994] use neural networks to recover an approximation of the Black-Scholes formula from simulated option prices and to fit observed prices of options on S&P 500 futures. Neural networks are non-linear regression methods whose functional form is very flexible due to the multi-

layered use of simple functions such as  $f(x) = 1/(1 + e^{-x})$ . These functions are applied to the data, and then the results are weighted by the regression coefficients and added together; finally the functions are again applied to the sums. More recently, Herrmann and Narr [1997] use the same methodology on German index option prices, which exhibit a volatility smile.

#### IV. CONCLUSION

Since the crash of 1987, researchers have realized that the Black-Scholes model can no longer explain the observed volatility smile in the index options markets. Options on individual stocks and exchange rates also exhibit volatility smiles. These conditions have led to renewed interest in parametric extensions of the Black-Scholes model that incorporate stochastic volatility and stochastic jumps.

An alternative approach is to use observed option prices in order to learn more about the stochastic process of the asset price. Given a set of option prices with specific times to expiration, we can find risk-neutral probability distributions that support these prices. A number of studies assess the change in these probability distributions due to news events.

Furthermore, we can also learn from observed option prices about the stochastic process of the asset price, leading up to the terminal probability distribution. Implied binomial trees allow us to recover such stochastic processes, and extensions are available that can incorporate stochastic volatility processes as well. In empirical tests, implied binomial trees perform as well (or as poorly) as both parametric models and naive trader rules (such as the absolute and relative smile rules).

#### ENDNOTE

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